THE SOBOLEV GRADIENT REGULARIZATION STRATEGY FOR OPTIMAL TOMOGRAPHY COUPLED WITH A FINITE ELEMENT FORMULATION OF THE RADIATIVE TRANSFER EQUATION

Fabien Dubot, Olivier Balima, Yann Favennec, Daniel Rousse

LTN UMR CNRS 6607 and Chaire T3E
University of Nantes and ÉTS, Bourget Lévis, Canada
yann.favennec@univ-nantes.fr

ABSTRACT

The inverse problem of optical tomography is ill-posed. Usually, the zero–order Tihonov regularization penalization is used. Though this method may be efficient in special circumstances, the use of the Sobolev gradient is highly more efficient.

1. INTRODUCTION

In optical tomography, the optical properties of the medium under investigation are obtained through the solution of an inverse problem where some light is injected on some boundaries and the measurement is performed elsewhere on the boundary in terms of light intensity [1, 2]. Often, the properties of interest are the scaterring and the diffusion coefficients, denoted as $\sigma(\boldsymbol{x})$ and $\kappa(\boldsymbol{x})$ where in this case $\boldsymbol{x}$ is in a two dimensional region. Such an inverse problem is solved minimizing a $L_2(\partial\Omega)$ norm between the prediction and the measurement. It is well known that such inverse problem is ill-posed, thus regularization must be used. In this talk, the regularization is based on the so-called Sobolev-gradient since this one acts as a pre-conditionner within the optimization algorithm while smooting the cost function gradient which contains the noise due to experimental set-up.

2. FORWARD MODEL

The forward model used is this study is the frequency domain form of the radiative transfer equation which consists in the following intro-differential equation [1, 3]:

$$\left(\vec{\Omega}\cdot\nabla + \frac{i\omega}{c} + \kappa + \sigma\right) I(\boldsymbol{x},\vec{\Omega},\omega) = \frac{\sigma}{4\pi}\int_{4\pi} I(\boldsymbol{x},\vec{\Omega}',\omega)\Phi(\vec{\Omega}',\vec{\Omega})\,\mathrm{d}\Omega' \tag{1}$$

where $\vec{\Omega}$ is the propagation direction of light, I is the radiant power per unit solid angle per unit area at spatial position $\boldsymbol{x}$ in direction $\vec{\Omega}$, and κ and σ are the absorption and scattering coefficients that depend on $\boldsymbol{x}$ and that have to be retrieved from measurements. $\Phi(\vec{\Omega}',\vec{\Omega})$ is the scattering phase function. Usually, the scattering in tissues is described by the Henyey-Greenstein phase function [4].

The chosen strategy consists in separating the complex intensity involved in the radiative transfer equation (1) into the collimated and the scattered intensity. One has then to couple with two coupled "advection-like" equations. The integral $\int_{4\pi}\cdot\,\mathrm{d}\Omega'$ is approached using the discrete ordinates method [5]. The used numerical scheme is the least square finite element formulation described in [6]. The numerical developments are integrated with the FreeFem++ environment [7].

3. INVERSE PROBLEM

In optical tomography, the cost function is generally expressed as errors between measurements and prediction. The cost function is written as:

$$\mathcal{J}(I) = \frac{1}{2}\sum\!\!\!\!\!\!\int \|P - M\|^2 \tag{2}$$

where the symbol $\sum\!\!\!\!\!\!\int$ stands for either discrete or continuous integration of discrepancies on the boundary. Assuming that κ and σ belong to the same finite element space, and $\theta = (\kappa,\sigma) \in \mathbf{R}^{2N_c}$ where N_c is the number of degrees of freedom of the chosen finite element space, then, one has to minimize $j(\theta) := \mathcal{J}(I)$.

It can be shown that the directional derivative of the cost $j(\theta)$ in feasible directions κ' and σ' is given by:

$$j'(\theta;\kappa') = \langle \kappa' I_c, I_c^* \rangle_{L_2(\mathcal{D})} + \langle \kappa' I_s, I_s^* \rangle_{L_2(\mathcal{D})} \qquad (3)$$

$$\begin{aligned} j'(\theta;\sigma') &= \langle \sigma' I_c, I_c^* \rangle_{L_2(\mathcal{D})} + \langle \sigma' I_s, I_s^* \rangle_{L_2(\mathcal{D})} \\ &- \Big\langle \frac{\sigma'}{4\pi} \int_{4\pi} \Big[I_s(r, \vec{\Omega}', \omega) + I_c(r,\omega)\delta(\vec{\Omega}' - \vec{\Omega}_c) \Big] \\ &\Phi(\vec{\Omega}', \vec{\Omega})\,\mathrm{d}\Omega', I_s^* \Big\rangle_{L_2(\mathcal{D})} \end{aligned} \qquad (4)$$

where I_s^* and I_c^* are the adjoint (co-states) versions of the states I_s and I_c with the inner product $\langle u, v \rangle_{L_2(\mathcal{D})} := \int_{\mathcal{D}} \bar{u} v \,\mathrm{d}\boldsymbol{x}$.

Most often, the cost function gradient is extracted from the classical inner product $j'(\theta;\theta') = (\nabla j, \theta')_{L_2(\mathcal{D})}$ where $(u,v)_{L_2(\mathcal{D})} := \int_{\mathcal{D}} \bar{u} v \,\mathrm{d}x$ and gradient-like optimization algorithms such as the limited memory BFGS (L–BFGS) is used [8]. However, the inverse problem being ill-posed by nature, some regularization tools must be performed. The Tikhonov penalization is the most commonly used strategy for regularization in optimal tomography. We here present another approach that consists in filtering spacially the noise existing in the cost gradient due to the presence of noise, inherently, from the measurements. The employed solution consists in using the so-called Sobolev gradient that is very much used in image segmentation. It is also shown in [9] that the use of such following inner product (where ℓ is a "tuning" parameter) for extracting the gradient acts as smooting the gradient, and also acts as a preconditionner for the optimization problem:

$$(u,v)_{H^{1(\ell)}(\mathcal{D})} := \frac{1}{1+\ell^2} \int_{\mathcal{D}} \left(\bar{u} v + \ell^2 \nabla \bar{u} \cdot \nabla v \right) \mathrm{d}x \qquad (5)$$

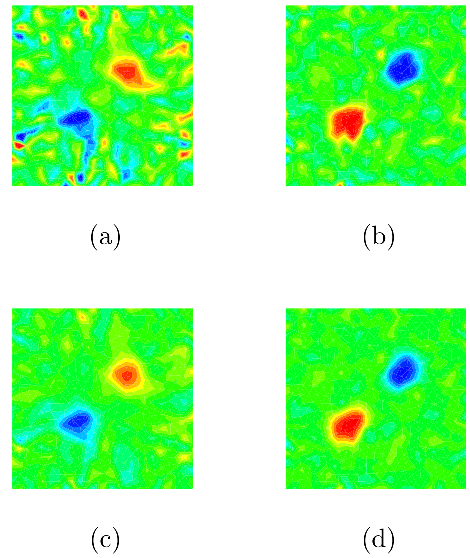

Figure 1: Two inclusions reconstruction with -20 dB noisy data. Recovered distribution of the absorption (a)-(c) and reduced scattering coefficient (b)-(d) with the $L_2(\mathcal{D})$ inner product (a)-(b) and with the $H^{1(\ell)}(\mathcal{D})$ inner product (c)-(d).

4. REFERENCES

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