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Investigation of angular discretization schemes for the Radiative Tranfer Equation

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Abstract

The present paper deals with the solution of the Radiative Transfer Equation (RTE) which is an integro-differential equation. In conjonction with the discontinuous Galerkin method, several methods are considered in order to cope with the angular integration term. These methods are formalized mathematically and then compared numerically. Especially, a non-uniform lagrangian basis is introduced.

Key words: Discontinuous Galerkin, ETR, FEM, Lagrangian basis

1 Introduction

The forward model commonly used to model the propagation of light within a heterogeneous semi-transparent medium is the so-called "Radiative Transfer Equation" (in short RTE). This is an integro-differential Boltzmann-type equation. Written in the frequency domain, the RTE problem consists in searching the radiance $L(\boldsymbol{x},\boldsymbol{s})$ such that:

$$\left(\boldsymbol{s}\cdot\nabla+\frac{i\omega}{c}+\mu_a+\mu_s\right)L(\boldsymbol{x},\boldsymbol{s})-\mu_s\oint_{\mathcal{S}^{n-1}}L\Phi(\boldsymbol{s},\boldsymbol{s}')\,\mathrm{d}\boldsymbol{s}'=0\quad\forall(\boldsymbol{x},\boldsymbol{s})\in\mathcal{D}\times\mathcal{S}^{n-1}\qquad(1)$$

where $\boldsymbol{s}\in\mathcal{S}^{n-1}$ in (1) is the direction of propagation of L at the location $\boldsymbol{x}$ ($\mathcal{S}^{n-1}$ is the unit disk in 2D and the unit sphere in 3D); μ_a and μ_s are the homogeneized absorption and scattering coefficients, respectively; ω is the modulation frequency, c is the speed of light and $i=\sqrt{-1}$. Physically, the radiance $L(\boldsymbol{x},\boldsymbol{s})$ is the radiant power per unit solid angle and per unit area; L depends on both the location $\boldsymbol{x}$ and the direction of propagation $\boldsymbol{s}$. Next, $\Phi(\boldsymbol{s},\boldsymbol{s}')$ is the probability density scattering phase function which is assumed herein to follow the HenyeyâĂŞGreenstein formulation. In applications considered here, the medium is illuminated with a collimated beam of radiation (i.e. with $\boldsymbol{s}=\boldsymbol{s}_0$, and $\boldsymbol{s}\cdot\boldsymbol{n}<0$) on the surface boundary $\boldsymbol{x}\in\partial\mathcal{D}_0$, this Dirichlet boundary condition is written as $L=L_0(\boldsymbol{x})$.

The spatial discretization is based on the discontinuous Galerkin finite element method, this method being much less sensitive to attenuation than other stable least-squares-type methods for instance [1, 2]. This paper investigates several schemes for angular discretization for the integral $\oint_{\mathcal{S}^{n-1}}\cdot\,\mathrm{d}\boldsymbol{s}'$ involved in (1).

In most situations, equation (1) cannot be analytically solved due to the integral term $\oint_{\mathcal{S}^{n-1}} \cdot \, \mathrm{d}\boldsymbol{s}'$. To cope with such a difficulty, this integral is approximated by [3]:

$$\oint_{\mathcal{S}^{n-1}} L\Phi(\boldsymbol{s},\boldsymbol{s}') \, \mathrm{d}\boldsymbol{s}' \approx \sum_{j=1}^{\boldsymbol{N}_t} \boldsymbol{w}_j L(\boldsymbol{x},\boldsymbol{s}_j) \tag{2}$$

where $\boldsymbol{s}_j$ is the j^{th} selected direction and $\boldsymbol{w}_j$ is the associated weight. Several approximation methods are formalized and numerically compared in this paper: the classical quadrature, the gaussian quadrature, and the use of a lagrangian basis.

Classical quadratures are first considered: the integral can be approximated by the methods of rectangles or trapezia. These methods are simple to implement and allow to compare the efficiency of other complex methods.

The Gaussian quadrature, rather than being uniform, is based on a family of orthogonal polynoms $\{P_N(\boldsymbol{s})\}_N$ with associated weighting function $\omega(\boldsymbol{s})$. It is well accepted that this quadrature yields to much more accurate integral approximation than classical quadratures since it is exact for polynoms up to order $2\boldsymbol{N}_t - 1$ [4]. The $\boldsymbol{N}_t$ directions are the roots of $P_{\boldsymbol{N}_t}(\boldsymbol{s})$, and the weights $\boldsymbol{w}_j$ are given by:

$$\boldsymbol{w}_j = \frac{A_{\boldsymbol{N}_t+1}\gamma_{\boldsymbol{N}_t}}{A_{\boldsymbol{N}_t}P'_{\boldsymbol{N}_t}(\boldsymbol{s}_j)P_{\boldsymbol{N}_t+1}(\boldsymbol{s}_j)}\Phi(\boldsymbol{s},\boldsymbol{s}_j) \tag{3}$$

where $\gamma_{\boldsymbol{N}_t} = \int_{-\pi}^{\pi} \omega(\boldsymbol{s}) P^2_{\boldsymbol{N}_t}(\boldsymbol{s}) \, \mathrm{d}\boldsymbol{s}$ and $A_{\boldsymbol{N}_t}$ is the coefficient of $\boldsymbol{s}^{\boldsymbol{N}_t}$ in $P_{\boldsymbol{N}_t}(\boldsymbol{s})$.

The Lagrangian basis $\{Lg_m(\boldsymbol{s})\}_{1\leq m\leq \boldsymbol{N}_t}$ is used with $Lg_m(\boldsymbol{s})$ being a continuous and piecewise linear function ($\mathbb{P}_1$) satisfying $Lg_m(\boldsymbol{s}_j) = \delta_{jm}$. With such a basis, the radiance $L(\boldsymbol{x},\boldsymbol{s})$ can be approximated through $L(\boldsymbol{x},\boldsymbol{s}) \approx \sum_{j=1}^{\boldsymbol{N}_t} L(\boldsymbol{x},\boldsymbol{s}_j) Lg_j(\boldsymbol{s})$. The scattering function $\Phi(\boldsymbol{s},\boldsymbol{s}')$ is also approximated to $\tilde{\Phi}(\boldsymbol{s},\boldsymbol{s}')$, and the weights $\boldsymbol{w}_j$ are finally given by:

$$\boldsymbol{w}_j \approx \oint_{\mathcal{S}^{n-1}} Lg_j(\boldsymbol{s}')\tilde{\Phi}(\boldsymbol{s},\boldsymbol{s}') \, \mathrm{d}\boldsymbol{s}' \tag{4}$$

Though this method is mostly used based on a uniform angular discretization [1, 5], this paper discusses non-uniform discretizations. Also, this paper discusses advantages and disadvantages of each of the three presented methods for solving the RTE.

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